OCR A2 Mathematics (MEI)
C3 Coursework
Numerical Solutions of Equations

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Abstract

In this coursework I will investigate three methods of solving equations numerically:

1. *change of sign* by *decimal search*

2. use of the *Newton-Raphson* formula

3. rearrangement of $f(x) = 0$ in the form $x = g(x)$

1 Change of sign by decimal search

The equation I will solve is $f(x) = 0$ where $f(x) = 2x^3 - 4x + 1$. The graph of $f(x)$ is shown below.
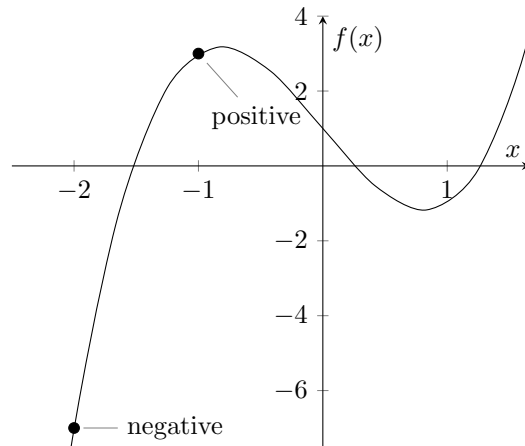

Figure 1: Plot of $f(x) = 2x^3 - 4x + 1$

As can be seen, at $x = -2$ the function is negative, but at $x = -1$ the function is positive. This *change of sign* indicates that a root to the function lies in the interval $[-2, -1]$.
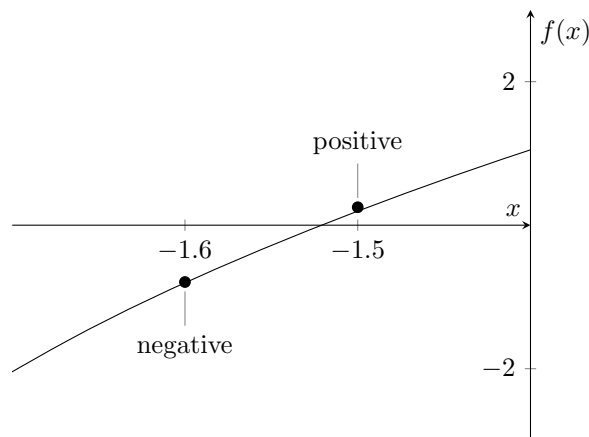

Figure 2: Plot of $f(x) = 2x^3 - 4x + 1$ highlighting $[-1.6, -1.5]$ interval

I will be finding the root which lies between $x = -2$ and $x = -1$ using the decimal search method. By further dividing the space in which we know the root lies, we can narrow it down further and discover that the root is between $x = -1.6$ and $x = -1.5$. One must then divide this new space again into 10.
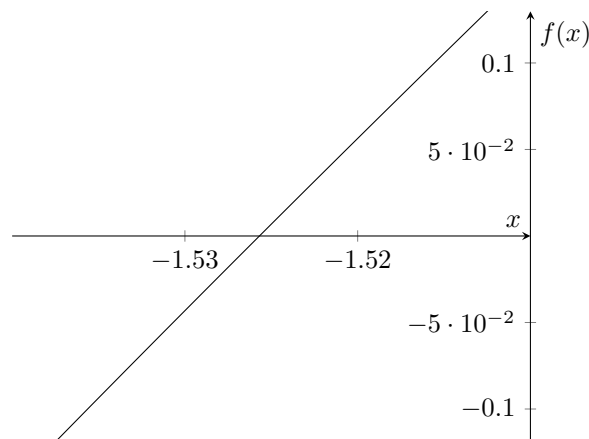

Figure 3: Plot of $f(x) = 2x^3 - 4x + 1$ highlighting $[-1.53, -1.52]$ interval
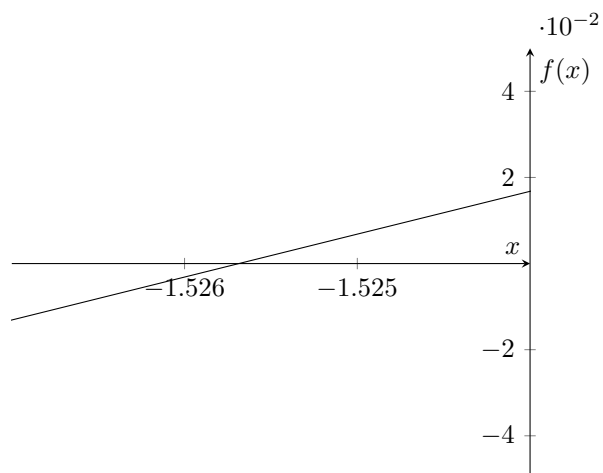
Figure 4: Plot of $f(x) = 2x^3 - 4x + 1$ highlighting $[-1.526, -1.525]$ interval
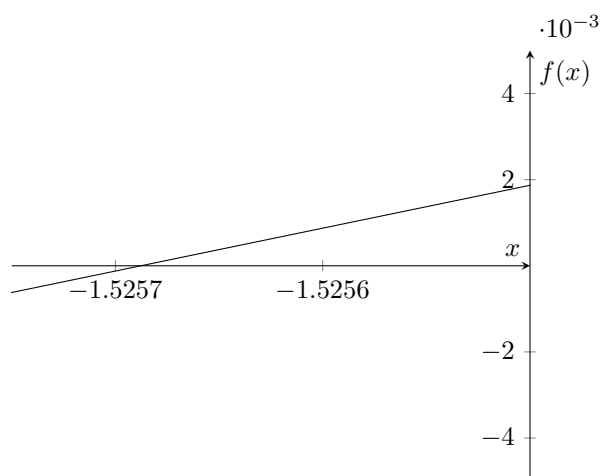

Figure 5: Plot of $f(x) = 2x^3 - 4x + 1$ highlighting $[-1.5257, -1.5256]$ interval

As can be seen in figures 3, 4 and 5, by going into greater detail with each iteration we can increase the number of decimal places to which we know the root. Thus, knowing that the root lies between $x = -1.5257$ and $x = -1.5256$ we can say that the root of $f(x) = 2x^3 - 4x + 1$ lying between $x = -2$ and $x = -1$ is -1.52565 ± 0.00005 at this fifth iteration.

However, as can be seen in figure 6 overleaf, having gone to eight iterations of the method we can find the root to a greater degree of accuracy: $-1.525687125 \pm 0.000000005$.

Figure 6 – Decimal Search Method: finding the 'change of sign' for the graph $y=2x^3-4x+1$ using Microsoft Excel

Yellow highlighting denotes where the change of sign occurs.

#												
1	x	-2	-1.9	-1.8	-1.7	-1.6	-1.5	-1.4	-1.3	-1.2	-1.1	-1
	y	-7	-5.118	-3.464	-2.026	-0.792	0.25	1.112	1.806	2.344	2.738	3
2	x	-1.6	-1.59	-1.58	-1.57	-1.56	-1.55	-1.54	-1.53	-1.52	-1.51	-1.5
	y	-0.792	-0.679358	-0.568624	-0.459786	-0.352832	-0.24775	-0.144528	-0.043154	0.056384	0.154098	0.25
3	x	-1.53	-1.529	-1.528	-1.527	-1.526	-1.525	-1.524	-1.523	-1.522	-1.521	-1.52
	y	-0.043154	-0.033117778	-0.023099904	-0.013100366	-0.003119152	0.00684375	0.016788352	0.026714666	0.036622704	0.046512478	0.056384
4	x	-1.526	-1.5259	-1.5258	-1.5257	-1.5256	-1.5255	-1.5254	-1.5253	-1.5252	-1.5251	-1.525
	y	-0.003119152	-0.002122038	-0.001125107	-0.000128359	0.000868206	0.001864587	0.002860786	0.003856801	0.004852634	0.005848283	0.00684375
5	x	-1.52569	-1.52568	-1.52567	-1.52566	-1.52565	-1.52564	-1.52563	-1.52562	-1.52561	-1.5256	-1.52559
	y	-2.86945E-05	7.09684E-05	0.000170629	0.000270289	0.000369946	0.000469602	0.000569255	0.000668907	0.000768557	0.000868206	0.000967852
6	x	-1.525689	-1.525688	-1.525687	-1.525686	-1.525685	-1.525684	-1.525683	-1.525682	-1.525681	-1.52568	-1.525679
	y	-1.87281E-05	-8.76175E-06	1.20459E-06	1.11709E-05	2.11372E-05	3.11035E-05	4.10697E-05	5.1036E-05	6.10022E-05	7.09684E-05	8.09346E-05
7	x	-1.5256879	-1.5256878	-1.5256877	-1.5256876	-1.5256875	-1.5256874	-1.5256873	-1.5256872	-1.5256871	-1.525687	-1.5256869
	y	-7.76511E-06	-6.76848E-06	-5.77185E-06	-4.77521E-06	-3.77858E-06	-2.78195E-06	-1.78531E-06	-7.8868E-07	2.07953E-07	1.20459E-06	2.20122E-06
8	x	-1.52568719	-1.52568718	-1.52568717	-1.52568716	-1.52568715	-1.52568714	-1.52568713	-1.52568712	-1.52568711	-1.5256871	-1.52568709
	y	-6.89017E-07	-5.89354E-07	-4.8969E-07	-3.90027E-07	-2.90364E-07	-1.90701E-07	-9.10372E-08	8.62604E-09	1.08289E-07	2.07953E-07	3.07616E-07

Figure 6: Change of sign in tabular form

1.1 Where does the decimal search method fail to find a root?

One example of failure for the decimal search method is where a function is asymptotic or discontinuous. This is because the method searches for a 'change of sign'. The change of sign would usually be seen where the root is, as the function goes from positive to negative or vice-versa, however with a discontinuous function this change of sign instead appears between the two vertical elements of the asymptotes, giving a *false root*.

$$f(x) = \frac{1}{2x^3 - 4x + 1}$$

An example of an equation for which this is the case is $\frac{1}{2x^3-4x+1} = 0$. One might find a false root at $x = -1.5$
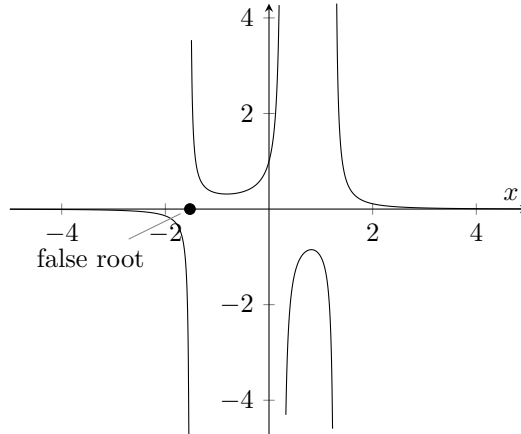

Figure 7: Plot of $f(x) = \frac{1}{2x^3-4x+1}$

where there is a change of sign due to the presence of an asymptote.

$$f(-2) = -0.142$$

$$f(-1) = \frac{1}{3}$$

As can be seen, a change of sign occurs between $x = -2$ and $x = -1$, which would lead us to believe that a root exists in the interval $[-2, -1]$, but this is not the case.

This error can be avoided by drawing an initial sketch of the graph and observing that it is in fact a discontinuous function and so may give incorrect results when the decimal search method is used. Looking at the values for $f(x)$ as we approach the false root from either side, it can also be observed that the gap between them widens rather than decreasing as it should - this would be an indicator that we were mistakenly finding the asymptote.

x	-4	-3	-2	-1	0	1	2	3	4	5	6
y	-0.009009009	-0.024390244	-0.142857143	0.333333333	1	-1	0.111111111	0.023255814	0.008849558	0.004329004	0.002444988

x	-2	-1.9	-1.8	-1.7	-1.6	-1.5	-1.4	-1.3	-1.2	-1.1	-1
y	-0.142857143	-0.195388824	-0.288683603	-0.493583416	-1.262626263	4	0.899280576	0.553709856	0.42662116	0.365230095	0.333333333

x	-1.6	-1.59	-1.58	-1.57	-1.56	-1.55	-1.54	-1.53	-1.52	-1.51	-1.5
y	-1.262626263	-1.471977956	-1.758631363	-2.174924856	-2.834210049	-4.036326942	-6.919074505	-23.17282291	17.73552781	6.48937689	4

x	-1.529	-1.528	-1.527	-1.526	-1.525	-1.524	-1.523	-1.522	-1.521	-1.52	-1.519
y	-30.1952625	-43.2902232	-76.33374518	-320.5999579	146.1187215	59.565108	37.43262222	27.30546603	21.49960705	17.73552781	15.09723784

Figure 8: Change of sign - finding a false root

As can be seen in figure 8, without doing an initial sketch, one would find a false root of -1.5255 ± 0.0005 as a 'change of sign' occurs here. Thus this is an example of failure of the decimal search method.

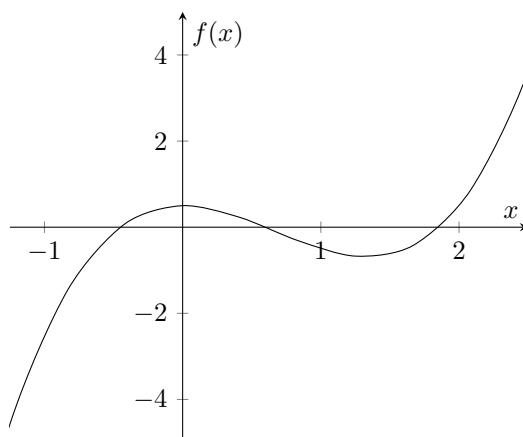# 2 The *Newton-Raphson* method

Figure 9: Plot of $f(x) = x^3 - 2x^2 + \frac{1}{2}$

I will use the *Newton-Raphson* method to find all three solutions to the equation $x^3 - 2x^2 + \frac{1}{2} = 0$, and shall illustrate the method to find the root lying between $x = 1$ and $x = 2$.

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

To use the *Newton-Raphson* method, we start with a value which is an estimate of the root (x_0). Putting this value into the formula we then gain a second value which should, assuming the formula works correctly, be a better estimate of the root. The output of this value is then put back into the formula and the exercise repeated - a 'fixed point iteration' method.

The formula works by calculating where the tangent to the curve at the first value crosses the x-axis, which is the second estimate, and then does the same with the tangent to the curve at this second value, and so on.

2.1 Finding the root in the range $1 < x < 2$

$$f(x) = x^3 - 2x^2 + \frac{1}{2}$$

$$f'(x) = 3x^2 - 4x$$

Having found $f'(x)$ we can then find the tangent at x_0 ($x = 2$), which can be modelled by $y = 4x - 7.5$.
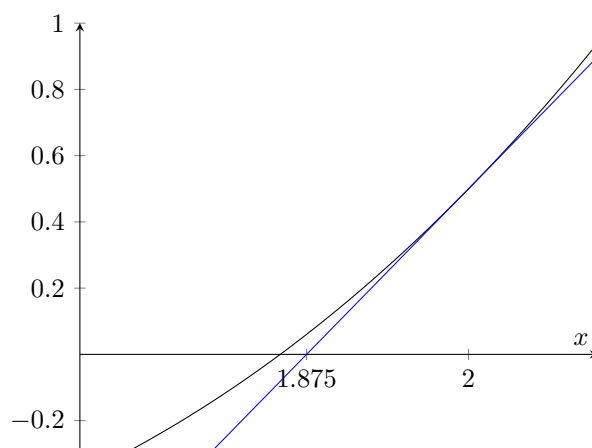

Figure 10: Tangent to $f(x)$ at x_0

In figure 10, we can see that the tangent at $x = 2$ crosses the x-axis at $x = 1.875$. Thus $x_1 = 1.875$, and this is put back into the formula, finding the tangent at $x = 1.875$. The tangent at x_1 is $y = 3.046875x - 5.65234375$, which crosses the x-axis at $x = 1.85512820$, which is the value for x_2
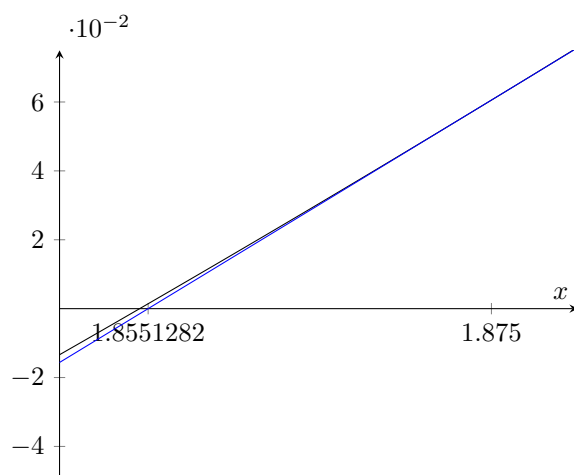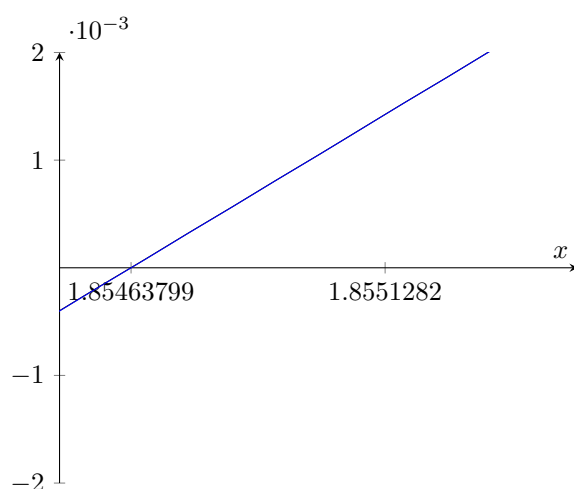
Figure 11: Tangent to $f(x)$ at x_1

Figure 12: Tangent to $f(x)$ at x_2

Iteration	Value of x (9sf)	Difference
x_0	2	
x_1	1.875	0.124999949
x_2	1.85512820	0.0198718328
x_3	1.85463798	0.00049024279
x_4	1.85463768	$2.95666394 \cdot 10^{-7}$
x_5	1.85463768	$2.73780998 \cdot 10^{-13}$

Table 1: *Newton-Raphson* iterations for $f(x) = x^3 - 2x^2 + \frac{1}{2}$ between $x = 1$ and $x = 2$

x_2 is then put back into the *Newton-Raphson* formula and the tangent to the curve at x_2 ($y = 2.903989115x - 5.385848492$) drawn. The point of intersection with the x-axis, $x_3 = 1.854637975$ (1.85463798 to 9 s.f.).

As can be seen in Table 1, using the *Newton-Raphson* formula we can find the root of $f(x) = x^3 - 2x^2 + \frac{1}{2}$ lying between $x = 1$ and $x = 2$ to be $1.85463768 \pm 0.000000005$. It took five iterations to find this root confidently to nine significant figures. Using a software package such as *Autograph* or even a spreadsheet suite, it would be fairly easy to automate this process to find roots quickly.

Establishing error bounds for this root

It is necessary to establish the error bounds for this root in addition to stating them. This can be done by checking a change of sign occurs between the lower and upper bound.

$$f(1.85463768 - 0.000000005) = f(1.854637675) = -1.368586 \cdot 10^{-8}$$

$$f(1.85463768 + 0.000000005) = f(1.854637685) = 1.531906 \cdot 10^{-8}$$

A change of sign does indeed occur between the lower and upper bound, and thus the error bounds have been established.

2.2 Finding the other roots of $f(x) = x^3 - 2x^2 + \frac{1}{2}$

As evidenced in Tables 2 and 3, we can also use the *Newton-Raphson* formula to find the other two root sof this function.

Iteration	Value of X (9sf)	Difference
x_0	1	
x_1	0.499999185	0.500000815
x_2	0.600000196	0.100001011
x_3	0.596969693	0.00303050231
x_4	0.596968283	$1.41016915 \cdot 10^{-6}$
x_5	0.596968283	$1.42874601 \cdot 10^{-12}$
Thus we can say the root of $f(x) = x^3 - 2x^2 + \frac{1}{2}$ lying between $x = 0$ and $x = 1$ is $0.596968283 \pm 0.0000000005$		

Table 2: *Newton-Raphson* iterations for $f(x) = x^3 - 2x^2 + \frac{1}{2}$ between $x = 0$ and $x = 1$

Iteration	Value of X (9sf)	Difference
x_0	-1	
x_1	-0.642857226	0.357142774
x_2	-0.487473797	0.155383429
x_3	-0.453261492	0.0342123053
x_4	-0.451609753	0.00165173958
x_5	-0.451605963	$3.7896531 \cdot 10^{-6}$
x_6	-0.451605963	$2.24792962 \cdot 10^{-11}$
Thus we can say the root of $f(x) = x^3 - 2x^2 + \frac{1}{2}$ lying between $x = -1$ and $x = 0$ is $0.451605963 \pm 0.0000000005$		

Table 3: *Newton-Raphson* iterations for $f(x) = x^3 - 2x^2 + \frac{1}{2}$ between $x = -1$ and $x = 0$

2.3 When does the *Newton-Raphson* method fail?
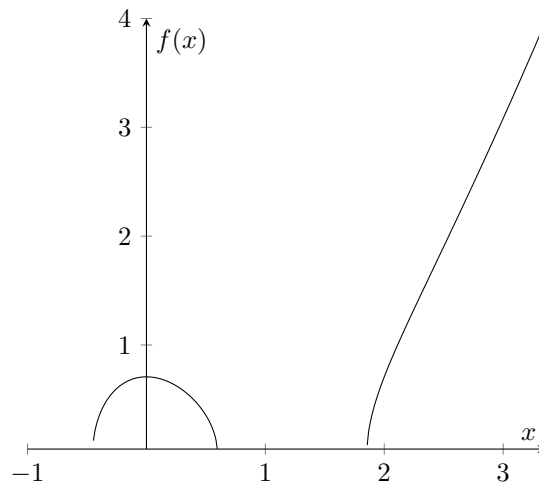
$$f(x) = \sqrt{x^3 - 2x^2 + \frac{1}{2}}$$

Figure 13: Plot of $f(x) = \sqrt{x^3 - 2x^2 + \frac{1}{2}}$

$$f'(x) = \frac{1.5(x - \frac{4}{3})x}{\sqrt{x^3 - 2x^2 + \frac{1}{2}}}$$

I will attempt to find the root of this function lying between $x = 0$ and $x = 1$. Taking a starting value of $x_0 = 0.5$, we find the tangent to the curve at this point to be $y = \frac{-5\sqrt{2}}{4}x + \frac{7\sqrt{2}}{8}$. This tangent crosses the x-axis
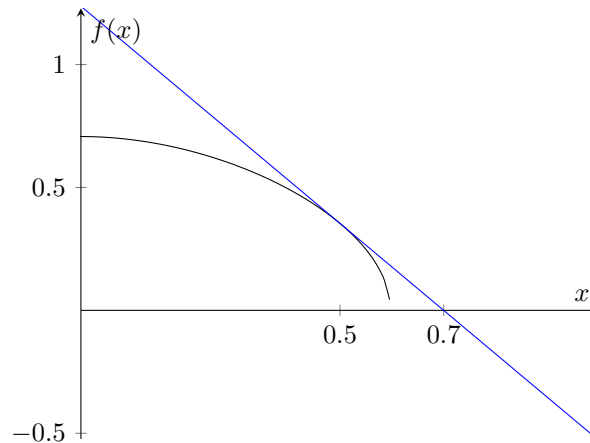

Figure 14: Tangent to $f(x) = \sqrt{x^3 - 2x^2 + \frac{1}{2}}$ at $x = 0.5$

at $x = 0.7$, thus $x_1 = 0.7$. However, when we put x_1 back into the *Newton-Raphson* formula, it fails as $f'(x)$ needs the square root of a negative, which is not possible.

$$f'(x) = \frac{1.5(x - \frac{4}{3})x}{\sqrt{x^3 - 2x^2 + \frac{1}{2}}}$$

$$f'(x_1) = \frac{1.5(0.7 - \frac{4}{3})0.7}{\sqrt{0.7^3 - 2(0.7)^2 + \frac{1}{2}}} = \frac{-0.665}{\sqrt{-0.137}}$$

Hence, a major limitation of the *Newton-Raphson* method is that one must be able to differentiate the function - which is problematic for some functions.

3 Rearrangement method

The equation I will solve is $f(x) = 0$ where $f(x) = 3x^3 - 4x + \frac{1}{2}$. The graph of $f(x)$ is shown below.
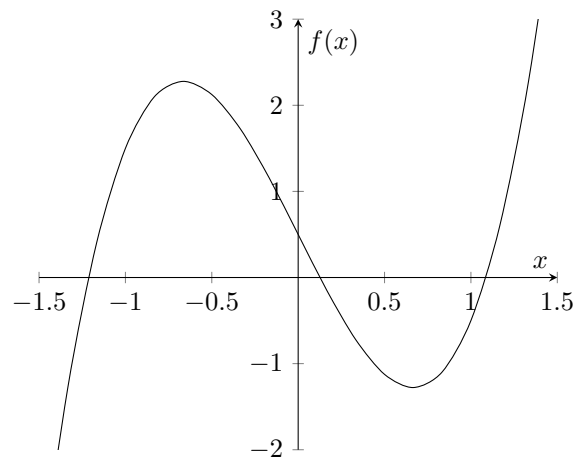

Figure 15: Plot of $f(x) = 3x^3 - 4x + \frac{1}{2}$

We must now make y equal to 0, then rearrange $f(x) = 0$ into $x = g(x)$. This produces a fixed-point iteration formula which we may use in a similar manner to the *Newton-Raphson* formula.

$$0 = 3x^3 - 4x + \frac{1}{2}$$

$$4x = 3x^3 + \frac{1}{2}$$

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$$x = \frac{3x^3 + \frac{1}{2}}{4}$$

$$x_{n+1} = \frac{3x_n^3 + \frac{1}{2}}{4}$$

By plotting $y = g(x)$, we can choose a suitable starting point to find our desired root.
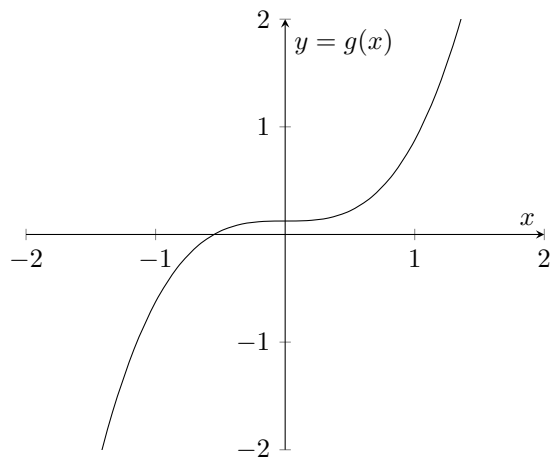

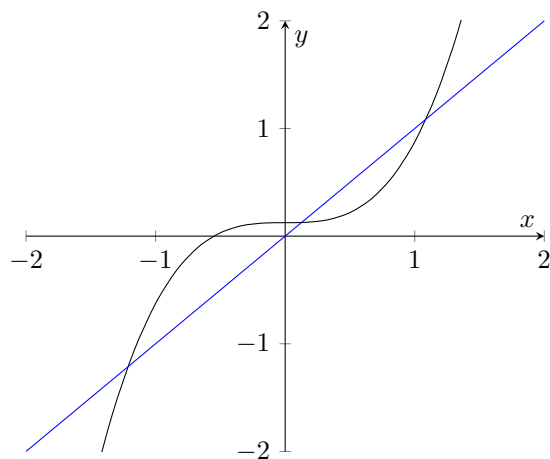Figure 16: Plot of $y = g(x)$ where $x = \frac{3x^3 + \frac{1}{2}}{4}$

Figure 17: Plot of $y = g(x)$ and $y = x$

By finding the points of intersection as in Figure 17, we can find the roots of $f(x)$. As can be seen in Figure 18 the x-values of the points of intersection of $g(x)$ and $y = x$ are the same as the roots of $f(x)$.
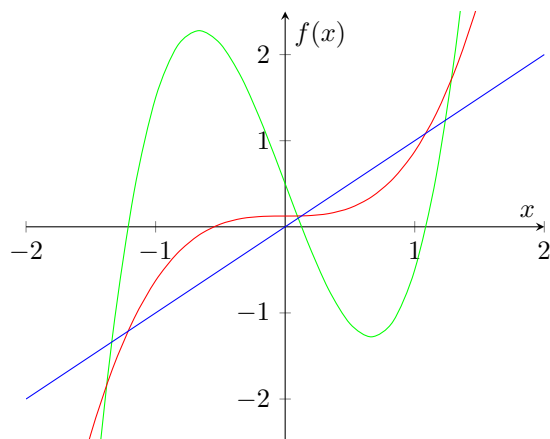

Figure 18: Plot of $f(x)$ (green), $g(x)$ (red) and $y = x$ (blue)

3.1 Finding the root in the range $x = 0$ and $x = 1$

I will use the rearrangement method to find the root of $f(x)$ lying between $x = 0$ and $x = 1$. As can be seen in 17 taking $x = 0$ as our starting value (x_0) is sensible. We can use the *Autograph* software package to carry out an $x = g(x)$ iteration with relative ease.

Figure 19: $x = g(x)$ iteration using *Autograph* for $f(x) = 3x^3 - 4x + \frac{1}{2}$ between $x = 0$ and $x = 1$

Figure 20: $x = g(x)$ iteration using *Autograph* for $f(x) = 3x^3 - 4x + \frac{1}{2}$ between $x = 0$ and $x = 1$
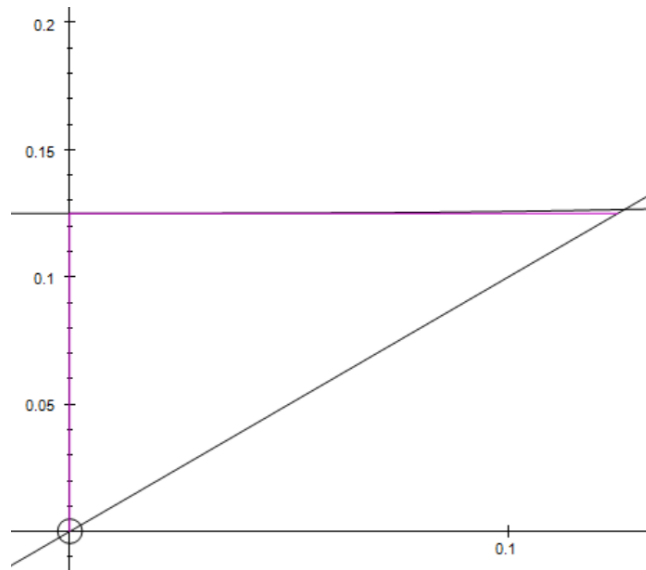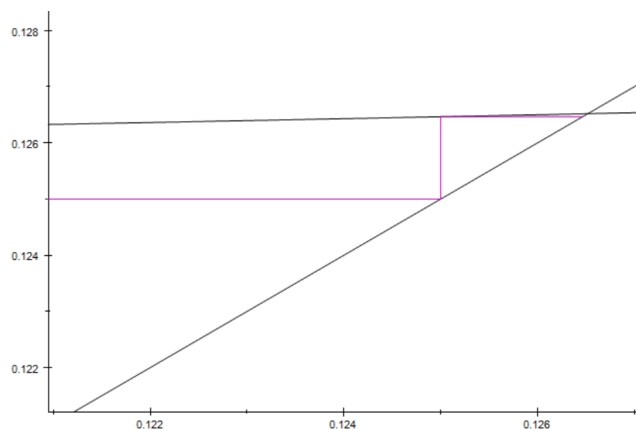
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Figure 21: $x = g(x)$ iteration using *Autograph* for $f(x) = 3x^3 - 4x + \frac{1}{2}$ between $x = 0$ and $x = 1$

Figure 22: $x = g(x)$ iteration using *Autograph* for $f(x) = 3x^3 - 4x + \frac{1}{2}$ between $x = 0$ and $x = 1$

Iteration	Value of x (9sf)
x_0	0
x_1	0.125
x_2	0.1264648438
x_3	0.1265169480
x_4	0.1265188238
x_5	0.1265188913
x_6	0.1265188938
x_7	0.1265188938
Thus we can say the root of $f(x) = 3x^3 - 4x + \frac{1}{2}$ lying between $x = 0$ and $x = 1$ is $0.1265188938 \pm 0.0000000005$	

Table 4: $x = g(x)$ iterations for $f(x) = 3x^3 - 4x + \frac{1}{2}$ between $x = 0$ and $x = 1$

3.2 $x = g(x)$ failure

I will now use the $x = g(x)$ iteration method to attempt to find the root of $f(x)$ lying between $x = 1$ and $x = 2$. As can be seen in 17, a starting value of $x_0 = 1$ can be chosen.
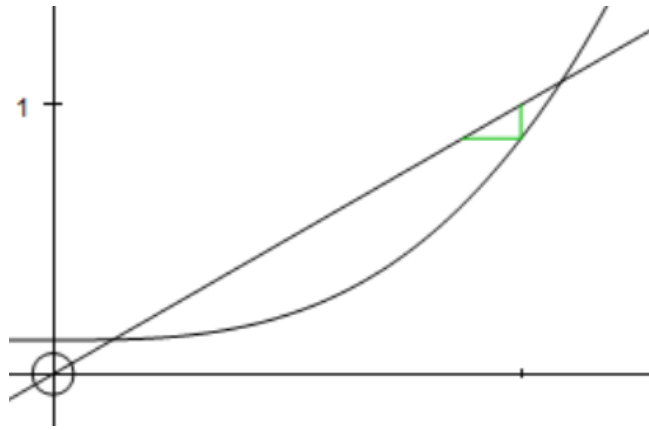
Figure 23: $x = g(x)$ iteration using *Autograph* for $f(x) = 3x^3 - 4x + \frac{1}{2}$ between $x = 0$ and $x = 1$
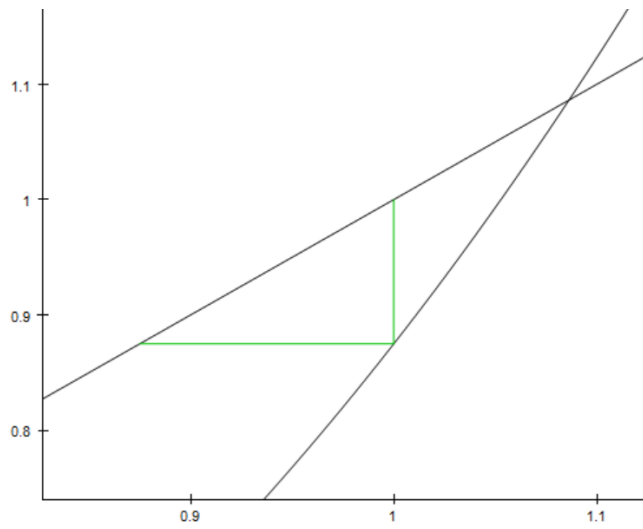
Figure 24: $x = g(x)$ iteration using *Autograph* for $f(x) = 3x^3 - 4x + \frac{1}{2}$ between $x = 0$ and $x = 1$
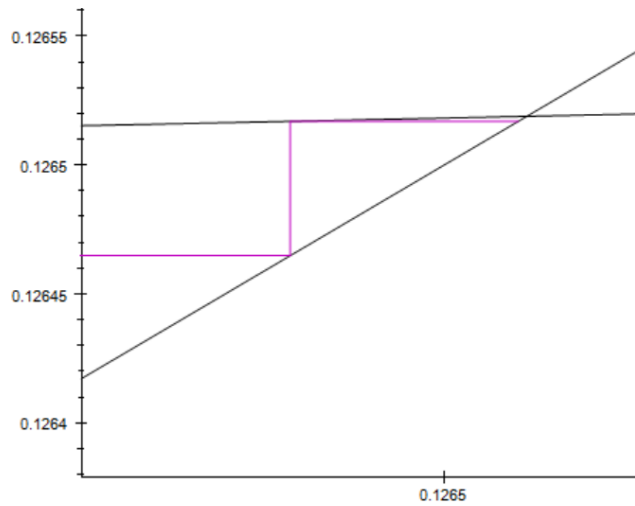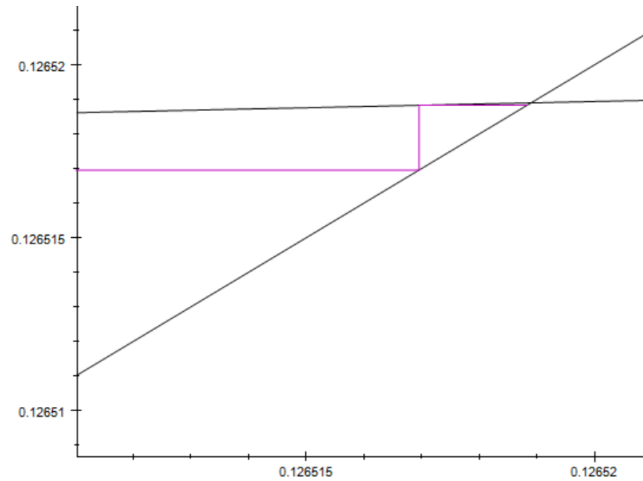

Figure 25: $x = g(x)$ iteration using *Autograph* for $f(x) = 3x^3 - 4x + \frac{1}{2}$ between $x = 0$ and $x = 1$

Iteration	Value of x
x_0	1
x_1	0.875
x_2	0.6274414063
x_3	0.3102596288
x_4	0.1473994353
$\vdots$	$\vdots$
x_{11}	0.1265188938

Table 5: $x = g(x)$ iterations for $f(x) = 3x^3 - 4x + \frac{1}{2}$ between $x = 1$ and $x = 2$

However, as can be seen, despite choosing a value of x_0 close to the root, the $x = g(x)$ iterations have found the wrong root - that closer to $x = 0$.

At $x = 1$ the tangent to $g(x)$ can be modelled by $y = 2.25000027x - 1.37500027$. Thus the gradient is 2.25. For the $x = g(x)$ method to work, $-1 < g'(x) < 1$ must be true - in this case the magnitude of $g'(x)$ is too great and so the method fails.

We can also attempt to find this root using a value the other side, for example, $x_0 = 1.5$.

Figure 26: $x = g(x)$ iteration using *Autograph* for $f(x) = 3x^3 - 4x + \frac{1}{2}$ between $x = 1$ and $x = 2$
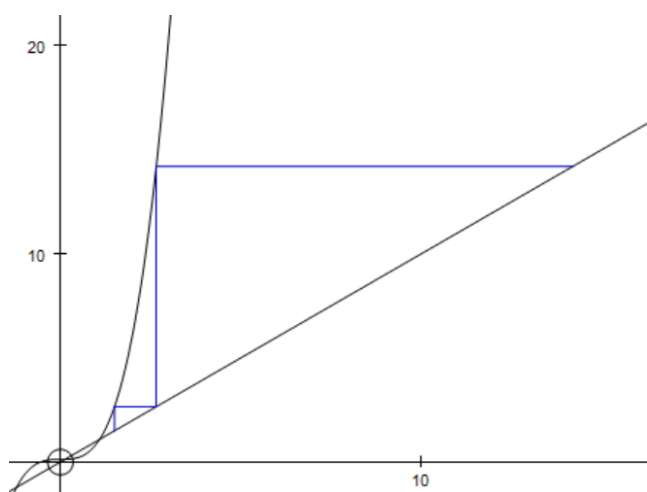
Figure 27: $x = g(x)$ iteration using *Autograph* for $f(x) = 3x^3 - 4x + \frac{1}{2}$ between $x = 1$ and $x = 2$
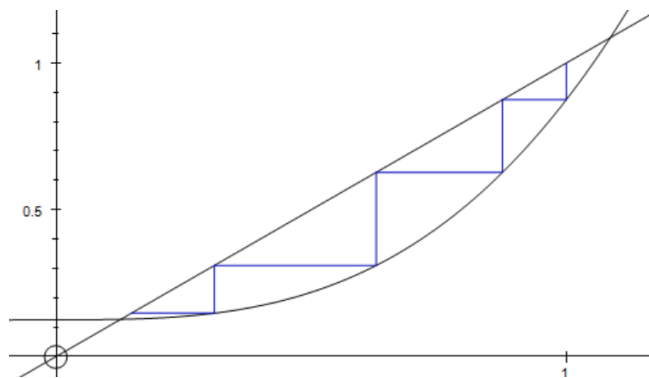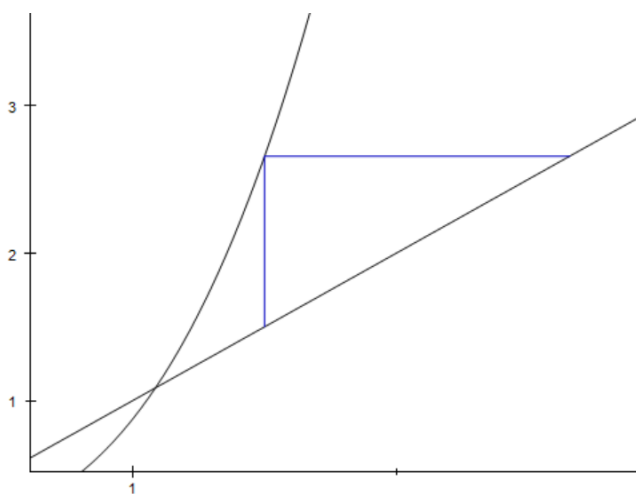
Iteration	Value of x
x_0	1.5
x_1	2.65625
x_2	14.18126575
x_3	2139.075517

Table 6: $x = g(x)$ iterations for $f(x) = 3x^3 - 4x + \frac{1}{2}$ between $x = 1$ and $x = 2$

As evidenced in Figures 26 & 27, the method still fails. The tangent to $g(x)$ at $x = 1.5$ is modelled by $y = 5.06250027x - 4.937500405$ - the magnitude of $g'(x)$ is again too great and so results in failure.

4   Comparison of methods

$$f(x) = 3x^3 - 4x + \frac{1}{2}$$

To compare the three methods of solving equations numerically, I will use each to find the same root of the equation $f(x) = 0$, where $f(x)$ is as above. I will specifically compare the number of iterations taken to find the root to nine significant figures. I will find the root lying between $x = 0$ and $x = 1$, with an x_0 of 0, or for the decimal search method, a starting range of $0 < x < 1$.

4.1   Decimal search method

As can be seen in Figure 28 overleaf, the decimal search method takes ten iterations to find our chosen root to nine significant figures.

#												
1	x	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1
	y	0.5	0.103	-0.276	-0.619	-0.908	-1.125	-1.252	-1.271	-1.164	-0.913	-0.5
2	x	0.1	0.11	0.12	0.13	0.14	0.15	0.16	0.17	0.18	0.19	0.2
	y	0.103	0.063993	0.025184	-0.013409	-0.051768	-0.089875	-0.127712	-0.165261	-0.202504	-0.239423	-0.276
3	x	0.12	0.121	0.122	0.123	0.124	0.125	0.126	0.127	0.128	0.129	0.13
	y	0.025184	0.021314683	0.017447544	0.013582601	0.009719872	0.005859375	0.002001128	-0.001854851	-0.005708544	-0.009559933	-0.013409
4	x	0.126	0.1261	0.1262	0.1263	0.1264	0.1265	0.1266	0.1267	0.1268	0.1269	0.127
	y	0.002001128	0.001615428	0.00122975	0.000844095	0.000458463	7.28539E-05	-0.000312733	-0.000698297	-0.001083838	-0.001469356	-0.001854851
5	x	0.1265	0.12651	0.12652	0.12653	0.12654	0.12655	0.12656	0.12657	0.12658	0.12659	0.1266
	y	7.28539E-05	3.42942E-05	-4.26526E-06	-4.28245E-05	-8.13835E-05	-0.000119942	-0.000158501	-0.000197059	-0.000235617	-0.000274175	-0.000312733
6	x	0.12651	0.126511	0.126512	0.126513	0.126514	0.126515	0.126516	0.126517	0.126518	0.126519	0.12652
	y	3.42942E-05	3.04382E-05	2.65823E-05	2.27263E-05	1.88704E-05	1.50144E-05	1.11585E-05	7.30255E-06	3.44661E-06	-4.09329E-07	-4.26526E-06
7	x	0.126518	0.1265181	0.1265182	0.1265183	0.1265184	0.1265185	0.1265186	0.1265187	0.1265188	0.1265189	0.126519
	y	3.44661E-06	3.06101E-06	2.67542E-06	2.28983E-06	1.90423E-06	1.51864E-06	1.13305E-06	7.47452E-07	3.61858E-07	-2.37356E-08	-4.09329E-07
8	x	0.1265188	0.12651881	0.12651882	0.12651883	0.12651884	0.12651885	0.12651886	0.12651887	0.12651888	0.12651889	0.1265189
	y	3.61858E-07	3.23299E-07	2.84739E-07	2.4618E-07	2.07621E-07	1.69061E-07	1.30502E-07	9.19425E-08	5.33832E-08	1.48238E-08	-2.37356E-08
9	x	0.12651889	0.126518891	0.126518892	0.126518893	0.126518894	0.126518895	0.126518896	0.126518897	0.126518898	0.126518899	0.1265189
	y	1.48238E-08	1.09679E-08	7.11192E-09	3.25599E-09	-5.99951E-10	-4.45589E-09	-8.31182E-09	-1.21678E-08	-1.60237E-08	-1.98796E-08	-2.37356E-08
10	x	0.126518893	0.126518893	0.126518893	0.126518893	0.126518893	0.126518894	0.126518894	0.126518894	0.126518894	0.126518894	0.126518894
	y	3.25599E-09	2.87039E-09	2.4848E-09	2.0992E-09	1.71361E-09	1.32802E-09	9.42424E-10	5.5683E-10	1.71236E-10	-2.14357E-10	-5.99951E-10

Figure 28: Change of sign in tabular form

4.2 *Newton-Raphson* formula

As seen, using the *Newton-Raphson* formula finds our root to nine significant figures in just four iterations.

Iteration	Value of x (9sf)	Difference
x_0	0	
x_1	0.125000036	0.125000036
x_2	0.126518219	0.00151818346
x_3	0.126518894	$6.74740823 \cdot 10^{-7}$
x_4	0.126518894	$6.5142336 \cdot 10^{-14}$
Thus we can say the root of $f(x) = 3x^3 - 4x + \frac{1}{2}$ lying between $x = 0$ and $x = 1$ is $0.126518894 \pm 0.0000000005$		

Table 7: *Newton-Raphson* iterations for $f(x) = 3x^3 - 4x + \frac{1}{2}$ between $x = 0$ and $x = 1$

4.3 Rearrangement method

Iteration	Value of x (9sf)
x_0	0
x_1	0.125
x_2	0.1264648438
x_3	0.1265169480
x_4	0.1265188238
x_5	0.1265188913
x_6	0.1265188938
x_7	0.1265188938
Thus we can say the root of $f(x) = 3x^3 - 4x + \frac{1}{2}$ lying between $x = 0$ and $x = 1$ is $0.1265188938 \pm 0.0000000005$	

Table 8: $x = g(x)$ iterations for $f(x) = 3x^3 - 4x + \frac{1}{2}$ between $x = 0$ and $x = 1$

As in Table 8, it takes seven iterations to find the root to nine significant figures.

4.4 Conclusion

In terms of speed of convergence, the *Newton-Raphson* method found the root to the desired level of accuracy most quickly (in the fewest iterations). However, one must also compare the ease-of-use of the three methods, and the software/hardware required to execute them.

The decimal search method is somewhat labour intensive if one wishes to find the root to any level of detail, as there is a lot of spreadsheet manipulation required. While the method itself is simple, regardless of the spreadsheet package used it can be tiresome to carry out. However, this method is fairly easy to understand and apply due to its simplicity. The prime benefit of the decimal search method is that can be done with simple software of which most people own a copy (eg *Microsoft Excel* is commonplace on most personal computers, and there are free and/or open-source alternatives such as *LibreOffice Calc* available on the Internet), or even with a simple scientific calculator. Indeed, it would be trivial to write a computer script to automate the decimal search method. The decimal search method was slowest to convergence, and there are also several potential issues with the method, in particular the possibility of finding a 'false root' should one not draw an initial sketch of the graph.

The *Newton-Raphson* formula, despite being the quickest in terms of number of iterations, does require an increased level of user understanding. It is the only method of the three that requires knowledge of differentiation, and so could cause issues if the user did not know how to differentiate. Indeed, a limitation is that we must also be able to differentiate the function of which we are seeking a root. Knowledge of differentiation is however not required if the user has access to a computer graphing package such as *Autograph* or *Microsoft Mathematics*. These applications will each carry out *Newton-Raphson* iterations autonomously, with the only user input required being the function, and a value of x_0. As long as the user has the necessary graphing software, this method was the quickest. The method can be used via a spreadsheet package, but this would require knowledge of differentiation - it would also be more labour-intensive. Thus, a downside of the *Newton-Raphson* method is the potential need for niche mathematics software. While free or open-source packages are available, the user would still be required to research and download them.

The $x = g(x)$ rearrangement method is simpler than the *Newton-Raphson* formula, requiring only the ability to rearrange $f(x) = 0$, which should be fairly easy for most functions. However, one does need to experiment to

ensure that a suitable rearrangement ($g(x)$) is selected. Like the *Newton-Raphson* formula, the rearrangement method can also be carried out using a spreadsheet package, but is far easier to carry out using a specialist mathematics package if available. In contrast to the *Newton-Raphson* formula, by not requiring knowledge of differentiation, it is a lot easier to carry out with a spreadsheet package than the *Newton-Raphson* formula. This method converged more quickly than the decimal search, but slower than the *Newton-Raphson* method. It should be noted that the speed of convergence using this method is directly related to the value of $g'(x)$ near the root: the closer to 0, the quicker the convergence. Hence the ability to differentiate is helpful when using the rearrangement method.

It is important to note that my experience of these methods of solving equations numerically is limited to this coursework, and so these conclusions will not necessarily apply to all cases.